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Sobolev inequality of free boundary value problem for $(-1)^M(d/dx)^{2M}$

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Abstract

The Green function of the free boundary value problem for $(-1)^M(d/dx)^{2M}$ is found by using Whipple's formula. The Green function is constructed through so-called symmetric orthogonalization method under a suitable solvability condition. Its Green function is a reproducing kernel for a suitable set of Hilbert space and an inner product. By using the fact, we compute the best constant ($M = 1, 2, 3, 4, 5$) and a family of the best functions for a Sobolev inequality. It is possible for us to expect the best constant of the Sobolev inequality, but the proof has not been completed for $M \geq 6$ in the present paper.

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1 Introduction

For $M = 1, 2, 3, \dots$, we introduce the Sobolev space

$$H = H(M) = \left\{ u(x) \mid u(x), u^{(M)}(x) \in L^2(-1, 1), \int_{-1}^1 u(x)x^i dx = 0 \ (0 \leq i \leq M-1) \right\}$$

with the Sobolev inner product

$$(u, v)_M = \int_{-1}^1 u^{(M)}(x)\overline{v^{(M)}(x)} dx.$$

$(\cdot, \cdot)_M$ is proven to be an inner product of H in Section 4. H is the Hilbert space with the inner product $(\cdot, \cdot)_M$. The purpose of the present paper is to find a supremum of the Sobolev functional given by

$$S(u) = S(M; u) = \left(\sup_{|y| \leq 1} |u(y)| \right)^2 / \|u\|_M^2, \quad \|u\|_M^2 = \int_{-1}^1 |u^{(M)}(x)|^2 dx.$$

The conclusion of the present paper is as follows.

Theorem 1.1 $G(x, y) = G(M; x, y)$ is the Green function which is defined later in Theorem 3.1. $\sup_{u \in H, u \neq 0} S(u) = C_0$ is given by

$$C_0 = C(M) = \max_{|y| \leq 1} G(y, y) = G(y_0, y_0) \quad (M \geq 1), \quad (1.1)$$

where y_0 attains the maximum value of the diagonal of the Green function. The supremum C_0 is attained by setting $u(x) = G(x, y_0)$. In particular, if $M \leq 5$, we have $y_0 = \pm 1$, in other words,

$$\max_{|y| \leq 1} G(y, y) = G(1, 1) = G(-1, -1) \quad (M = 1, 2, 3, 4, 5). \quad (1.2)$$

Explicit forms of $G(1, 1) = G(-1, -1)$ are given as

$$G(1, 1) = G(-1, -1) = \frac{2^{2M-1} \Gamma(2M-1) \Gamma(2M+1)}{\Gamma(M)^2 \Gamma(4M)} \quad (M \geq 1). \quad (1.3)$$

Note that the equality

$$\max_{|y| \leq 1} G(y, y) = G(\pm 1, \pm 1)$$

in (1.1) is proven only for $M \leq 5$ and is still open for $M \geq 6$. Here, we list explicit forms of $C(M)$ ($M = 1, 2, 3, 4, 5$),

$$\begin{aligned} C(1) &= \frac{2}{3}, & C(2) &= \frac{8}{105}, & C(3) &= \frac{4}{1,155}, \\ C(4) &= \frac{32}{405,405}, & C(5) &= \frac{4}{3,741,309}. \end{aligned}$$

Theorem 1.1 is equivalently rewritten as follows.

Theorem 1.2 *For any $u(x) \in H$, there exists a positive constant C such that the Sobolev inequality holds:*

$$\left(\sup_{|y| \leq 1} |u(y)| \right)^2 \leq C \int_{-1}^1 |u^{(M)}(x)|^2 dx.$$

Among such C , the best constant C_0 is the same as (1.1). If we replace C by C_0 , the equality holds for

$$u(x) = cG(x, y_0), \quad c \in \mathbb{C} \setminus \{0\} \quad (-1 < x < 1).$$

In particular, if $M \leq 5$, we have $y_0 = \pm 1$.

Concerning the infimum of $S(u)$, we have the following theorem.

Theorem 1.3 *The infimum of the Sobolev functional is equal to zero,*

$$\inf_{u \in H, u \neq 0} S(u) = 0. \quad (1.4)$$

Based on the previous research of Bliss [1], the best constant and a family of the best functions of the Sobolev inequality are first obtained independently by Aubin [2] and Talenti [3]. They mainly used the functional analysis technique to compute the best constant of the Sobolev inequality. On the other hand, we have computed the best constant

of the Sobolev inequality by using the Green function corresponding to a boundary value problem. This is a new method quite different from that of the functional analysis. The Green function is the reproducing kernel for suitable set of Hilbert space and inner product. As an application, the best constant of the corresponding Sobolev inequality is expressed as the maximum value of the diagonal of the Green function. References [4–6] are related to the early studies based on these facts. The engineering meaning of the Sobolev inequality becomes that the square of maximum bending of a string ($M = 1$) or a beam ($M = 2$) is estimated from above by the constant multiple of the potential energy [7, 8]. We have already obtained the best constant of each Sobolev inequality which corresponds to clamped-free, Dirichlet and periodic boundary value problems for $(-1)^M(d/dx)^{2M}$ [9–11]. Further, in [12], we consider a time-periodic boundary value problem of n th order ordinary differential operator which appears typically in Heaviside cable and Thomson cable theory. In this problem, the physical meaning of a Sobolev type inequality becomes that the square of maximum of the absolute value of AC output voltage is estimated above by the constant multiple of the power of input voltage. The purpose of the present paper is to derive a Sobolev inequality which corresponds to free boundary value problem for $(-1)^M(d/dx)^{2M}$ and obtain the best constant by using some properties as the reproducing kernel of the Green function.

This paper is organized as follows. In Section 2, we construct a proto Green function that becomes the origin of the Green function from the corresponding eigenvalue problem. In Section 3, we construct the Green function from the proto Green function as to satisfy the properties of reproducing kernel. We call the technique symmetric orthogonalization method [13]. In the method, Whipple's theorem concerning the hypergeometric series has an important role. In Section 4, we show that the Green function is the reproducing kernel for H and $(\cdot, \cdot)_M$. After deriving the Sobolev inequality, we give the proof of (1.1). Sections 5 and 6 are devoted to the proof of (1.2) ($M = 5$) and (1.3) in the main Theorem 1.1. In Section 7, we prove Theorem 1.3.

2 Free boundary value problem and the proto Green function

We start with the following lemma concerning the eigenvalue problem.

Lemma 2.1 *The eigenvalue problem*

$$\begin{cases} (-1)^M u^{(2M)} = \lambda u(x) & (-1 < x < 1), \\ u^{(i)}(\pm 1) = 0 & (M \leq i \leq 2M - 1) \end{cases}$$

has an eigenvalue $\lambda = 0$, and the corresponding eigenspace is M -dimensional. Its orthonormal base is

$$\left\{ \varphi_i(x) = \sqrt{i + \frac{1}{2}} P_i(x) \mid 0 \leq i \leq M - 1 \right\},$$

where $P_i(x)$ are Legendre polynomials.

We omit the proof of the lemma.

For convenience, we introduce the following monomials $\{K_j(x)\}$:

$$K_j(x) = K_j(M; x) = \begin{cases} \frac{x^{2M-1-j}}{\Gamma(2M-j)} & (0 \leq j \leq 2M-1), \\ 0 & (2M \leq j). \end{cases}$$

For any bounded continuous function $f(x)$ on $-1 < x < 1$ satisfying the solvability condition

$$\int_{-1}^1 f(y) \varphi_i(y) dy = 0 \quad (0 \leq i \leq M-1), \quad (2.1)$$

we consider the free boundary value problem

$$\text{BVP}(M) \quad \begin{cases} (-1)^M u^{(2M)} = f(x) & (-1 < x < 1), \\ u^{(i)}(\pm 1) = 0 & (M \leq i \leq 2M-1), \\ \int_{-1}^1 u(x) \varphi_i(x) dx = 0 & (0 \leq i \leq M-1). \end{cases} \quad (2.2)$$

$$(2.3)$$

$$(2.4)$$

We first introduce the proto Green function $G_0(x, y)$.

Lemma 2.2 *Suppose that the boundary value problem (2.2) and (2.3) has a classical solution $u(x)$. Then $f(x)$ satisfies the solvability condition (2.1), and $u(x)$ is expressed as*

$$u(x) = \sum_{i=0}^{M-1} \alpha_i \varphi_i(x) + \int_{-1}^1 G_0(x, y) f(y) dy \quad (-1 < x < 1), \quad (2.5)$$

where $\alpha_0, \alpha_1, \dots, \alpha_{M-1}$ are appropriate constants. $G_0(x, y)$ is a proto Green function or an equivalently fundamental solution for the differential operator $(-1)^M (d/dx)^{2M}$, which is defined by

$$G_0(x, y) = \frac{(-1)^M}{2} K_0(|x - y|) \quad (-1 < x, y < 1).$$

Proof We assume that (2.2) and (2.3) have a classical solution $u(x)$. Introducing new functions

$$\mathbf{u} = {}^t(u_0 \quad u_1 \quad \cdots \quad u_{2M-1}), \quad u_i = u^{(i)} \quad (0 \leq i \leq 2M-1)$$

and the $2M \times 2M$ nilpotent matrix

$$\mathbf{N} = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix},$$

we can rewrite (2.2) and (2.3) as

$$\mathbf{u}' = \mathbf{N}\mathbf{u} + {}^t(0 \quad \cdots \quad 0 \quad 1)(-1)^M f(x) \quad (-1 < x < 1), \quad (2.6)$$

$$u_i(\pm 1) = 0 \quad (M \leq i \leq 2M-1). \quad (2.7)$$

The fundamental solution $\mathbf{E}(x)$ to the above initial value problem is expressed as

$$\mathbf{E}(x) = \mathbf{K}(x)\mathbf{K}(0)^{-1},$$

where

$$\mathbf{K}(x) = (K_{i+j})(x) \quad (0 \leq i, j \leq 2M-1),$$

$$\mathbf{K}(0) = \begin{pmatrix} & & 1 \\ & \ddots & \\ 1 & & \end{pmatrix} = \mathbf{K}(0)^{-1}.$$

Solving (2.6), we have

$$\mathbf{u}(x) = \mathbf{E}(x+1)\mathbf{u}(-1) + \int_{-1}^x \mathbf{E}(x-y)^t (0 \quad \cdots \quad 0 \quad 1)(-1)^M f(y) dy,$$

$$\mathbf{u}(x) = \mathbf{E}(x-1)\mathbf{u}(1) - \int_x^1 \mathbf{E}(x-y)^t (0 \quad \cdots \quad 0 \quad 1)(-1)^M f(y) dy,$$

or equivalently, for $0 \leq i \leq 2M-1$,

$$u_i(x) = \sum_{j=0}^{2M-1} K_{i+j}(x+1)u_{2M-1-j}(-1) + \int_{-1}^x (-1)^M K_i(x-y)f(y) dy,$$

$$u_i(x) = \sum_{j=0}^{2M-1} K_{i+j}(x-1)u_{2M-1-j}(1) - \int_x^1 (-1)^M K_i(x-y)f(y) dy.$$

Employing the boundary conditions (2.7), we have

$$u_i(x) = \sum_{j=M}^{2M-1} K_{i+j}(x+1)u_{2M-1-j}(-1) + \int_{-1}^x (-1)^M K_i(x-y)f(y) dy, \quad (2.8)$$

$$u_i(x) = \sum_{j=M}^{2M-1} K_{i+j}(x-1)u_{2M-1-j}(1) - \int_x^1 (-1)^M K_i(x-y)f(y) dy \quad (2.9)$$

for $0 \leq i \leq 2M-1$. If $M \leq i \leq 2M-1$, we have

$$u_i(x) = \int_{-1}^x (-1)^M K_i(x-y)f(y) dy \quad (M \leq i \leq 2M-1),$$

$$u_i(x) = - \int_x^1 (-1)^M K_i(x-y)f(y) dy \quad (M \leq i \leq 2M-1).$$

Setting $x = \pm 1$ and employing the boundary conditions (2.7), we have

$$0 = u_i(1) = \int_{-1}^1 (-1)^M K_i(1-y)f(y) dy \quad (M \leq i \leq 2M-1), \quad (2.10)$$

$$0 = u_i(-1) = - \int_{-1}^1 (-1)^M K_i(-1-y)f(y) dy \quad (M \leq i \leq 2M-1). \quad (2.11)$$

From (2.10) and (2.11), it is shown that $f(x)$ satisfies the condition

$$\int_{-1}^1 y^i f(y) dy = 0 \quad (0 \leq i \leq M-1),$$

or equivalently

$$\int_{-1}^1 f(y) \varphi_i(y) dy = 0 \quad (0 \leq i \leq M-1),$$

which are necessary conditions for the existence of a classical solution to (2.2) and (2.3).

Setting $i = 0$ in (2.8) and (2.9), we have

$$\begin{aligned} u_0(x) &= \sum_{j=M}^{2M-1} K_j(x+1) u_{2M-1-j}(-1) + \int_{-1}^x (-1)^M K_0(x-y) f(y) dy, \\ u_0(x) &= \sum_{j=M}^{2M-1} K_j(x-1) u_{2M-1-j}(1) - \int_x^1 (-1)^M K_0(x-y) f(y) dy. \end{aligned}$$

Taking the average of the above two equalities, we obtain an expression for the solution

$$u(x) = u_0(x),$$

$$u(x) = \sum_{j=0}^{M-1} \alpha_j \varphi_j(x) + \int_{-1}^1 G_0(x, y) f(y) dy \quad (-1 < x < 1),$$

where α_j ($0 \leq j \leq M-1$) are suitable constants and

$$G_0(x, y) = \frac{(-1)^M}{2} K_0(|x-y|) \quad (-1 < x, y < 1).$$

This shows (2.5) in Lemma 2.2. \square

It is easy to prove the following lemma concerning the properties of the proto Green function $G_0(x, y)$. We omit the proof of the lemma.

Lemma 2.3 *The proto Green function $G_0(x, y)$ satisfies the following properties.*

- (1) $\partial_x^{2M} G_0(x, y) = 0 \quad (-1 < x, y < 1, x \neq y),$
- (2) $\partial_x^i G_0(x, y)|_{x=\pm 1} = (\pm 1)^i \frac{(-1)^M}{2} K_i(1 \mp y)$
 $(0 \leq i \leq 2M-1, -1 < y < 1),$
- (3) $\partial_x^i G_0(x, y)|_{y=x-0} - \partial_x^i G_0(x, y)|_{y=x+0} = \begin{cases} 0 & (0 \leq i \leq 2M-2), \\ (-1)^M & (i = 2M-1) \end{cases} \quad (-1 < x < 1),$
- (4) $\partial_x^i G_0(x, y)|_{x=y+0} - \partial_x^i G_0(x, y)|_{x=y-0} = \begin{cases} 0 & (0 \leq i \leq 2M-2), \\ (-1)^M & (i = 2M-1) \end{cases} \quad (-1 < y < 1).$

3 Method of symmetric orthogonalization

Concerning the uniqueness and existence of the solution to BVP(M), we have the following theorem.

Theorem 3.1 *For any bounded continuous function $f(x)$ on an interval $-1 < x < 1$ satisfying (2.1), BVP(M) has a unique classical solution $u(x)$ that is expressed as*

$$u(x) = \int_{-1}^1 G(x, y) f(y) dy \quad (-1 < x < 1).$$

The integral kernel $G(x, y) = G(M; x, y)$ is given by

$$G(x, y) = G_0(x, y) - \sum_{i=0}^{M-1} \{ \psi_i(x) \varphi_i(y) + \psi_i(y) \varphi_i(x) \} + \sum_{i,j=0}^{M-1} \gamma_{ij} \varphi_i(x) \varphi_j(y) \quad (-1 < x, y < 1), \quad (3.1)$$

where

$$\psi_i(x) = \int_{-1}^1 G_0(x, y) \varphi_i(y) dy \quad (0 \leq i \leq M-1, -1 < x < 1), \quad (3.2)$$

$$\gamma_{ij} = \int_{-1}^1 \psi_i(x) \varphi_j(x) dx = \int_{-1}^1 \varphi_i(x) \psi_j(x) dx = \gamma_{ji} \quad (0 \leq i, j \leq M-1). \quad (3.3)$$

The above procedure, in which the Green function $G(x, y)$ is constructed from the proto Green function $G_0(x, y)$, is called the symmetric orthogonalization method [5, 6, 13]. The proof of Theorem 3.1 is given in [13]. In the present paper, we give a more closed expression of $G(x, y)$.

We have the following theorem.

Theorem 3.2 *For $j = 0, 1, \dots, M-1$, the following equalities hold:*

$$\begin{aligned} \psi_j(x) &= \frac{(-1)^M}{2} \sqrt{j + \frac{1}{2}} \left\{ Q_j(x) + (-1)^j Q_j(-x) \right\}, \\ Q_j(x) &= \sum_{k=0}^j \frac{2^{2M} (-1)^{j+k} \Gamma(j+k+1)}{\Gamma(2M+k+1) \Gamma(k+1) \Gamma(j-k+1)} \left(\frac{1+x}{2} \right)^{2M+k} \quad (-1 < x < 1). \end{aligned} \quad (3.4)$$

Proof From (3.2) in Theorem 3.1, the functions $\psi_j(x)$ are calculated as

$$\begin{aligned} \psi_j(x) &= \int_{-1}^1 G_0(x, y) \varphi_j(y) dy = \frac{(-1)^M}{2} \sqrt{j + \frac{1}{2}} \int_{-1}^1 K_0(|x-y|) P_j(y) dy \\ &= \frac{(-1)^M}{2} \sqrt{j + \frac{1}{2}} \left\{ \int_{-1}^x K_0(x-y) P_j(y) dy + \int_x^1 K_0(y-x) P_j(y) dy \right\} \\ &= \frac{(-1)^M}{2} \sqrt{j + \frac{1}{2}} \left\{ \int_{-1}^x K_0(x-y) P_j(y) dy + \int_{-1}^{-x} K_0(-y-x) P_j(-y) dy \right\} \\ &= \frac{(-1)^M}{2} \sqrt{j + \frac{1}{2}} \left\{ Q_j(x) + (-1)^j Q_j(-x) \right\}, \end{aligned} \quad (3.5)$$

where

$$Q_j(x) = \int_{-1}^x K_0(x-y)P_j(y) dy \quad (-1 < x < 1).$$

It is shown through direct calculations that $u = Q_j(x)$ satisfies the initial value problem:

$$\begin{cases} u^{(2M)} = P_j(x) & (-1 < x < 1), \\ u^{(k)}(-1) = 0 & (0 \leq k \leq 2M-1). \end{cases} \quad (3.6)$$

$$(3.7)$$

The Legendre polynomial $P_j(x)$ is expressed as

$$P_j(x) = \sum_{k=0}^j \frac{(-1)^{j+k} \Gamma(j+k+1)}{\Gamma(k+1)^2 \Gamma(j-k+1)} \left(\frac{1+x}{2} \right)^k \quad (-1 < x < 1),$$

and integrating (3.6) $2M$ times under the initial conditions (3.7), we obtain (3.4). This proves Theorem 3.2. \square

Theorem 3.3 For $M = 1, 2, 3, \dots$, the coefficients γ_{ij} ($0 \leq i, j \leq M-1$) are expressed as

$$\gamma_{ij} = \begin{cases} \frac{(-1)^{j+M} \pi \Gamma(2M+1) \sqrt{i+\frac{1}{2}} \sqrt{j+\frac{1}{2}}}{2^{2M} \Gamma(\frac{2M-i-j+1}{2}) \Gamma(\frac{2M+i-j+2}{2}) \Gamma(\frac{2M-i+j+2}{2}) \Gamma(\frac{2M+i+j+3}{2})} & (i-j: \text{even}), \\ 0 & (i-j: \text{odd}). \end{cases}$$

Proof From (3.3) and (3.5), we have

$$\begin{aligned} \gamma_{ij} &= \int_{-1}^1 \varphi_i(x) \psi_j(x) dx \\ &= \frac{(-1)^M}{2} \sqrt{\left(i + \frac{1}{2}\right) \left(j + \frac{1}{2}\right)} \left\{ \int_{-1}^1 P_i(x) Q_j(x) dx + (-1)^j \int_{-1}^1 P_i(x) Q_j(-x) dx \right\} \\ &= \frac{(-1)^M}{2} \sqrt{\left(i + \frac{1}{2}\right) \left(j + \frac{1}{2}\right)} \{1 + (-1)^{i-j}\} \int_{-1}^1 P_i(x) Q_j(x) dx \\ &= \begin{cases} (-1)^M \sqrt{\left(i + \frac{1}{2}\right) \left(j + \frac{1}{2}\right)} \int_{-1}^1 P_i(x) Q_j(x) dx & (i-j: \text{even}), \\ 0 & (i-j: \text{odd}). \end{cases} \end{aligned}$$

Hereinafter, we investigate the case in which $i-j$ is even. From the Rodrigues formula and (3.4), we have

$$\begin{aligned} &\int_{-1}^1 P_i(x) Q_j(x) dx \\ &= \frac{1}{\Gamma(i+1)2^i} \sum_{k=0}^j \frac{(-1)^{j+k} \Gamma(j+k+1)}{\Gamma(2M+k+1) \Gamma(k+1) \Gamma(j-k+1) 2^k} \\ &\quad \times \int_{-1}^1 \left((-1)^i \left(\frac{d}{dx} \right)^i (1-x^2)^i \right) (1+x)^{2M+k} dx \\ &= \frac{1}{\Gamma(i+1)2^i} \sum_{k=0}^j \frac{(-1)^{j+k} \Gamma(j+k+1)}{\Gamma(2M+k+1) \Gamma(k+1) \Gamma(j-k+1) 2^k} \end{aligned}$$

$$\begin{aligned}
& \times (2M+k)(2M+k-1) \cdots (2M+k-i+1) \int_{-1}^1 (1-x^2)^i (1+x)^{2M+k-i} dx \\
& = \frac{1}{\Gamma(i+1)2^i} \sum_{k=0}^j \frac{(-1)^{j+k} \Gamma(j+k+1)}{\Gamma(2M+k-i+1) \Gamma(k+1) \Gamma(j-k+1) 2^k} \\
& \quad \times 2^{2M+k+i+1} B(i+1, 2M+k+1) \\
& = (-1)^j 2^{2M+1} \sum_{k=0}^j \frac{(-1)^k \Gamma(2M+k+1) \Gamma(j+k+1)}{\Gamma(2M+k-i+1) \Gamma(2M+k+i+2) \Gamma(k+1) \Gamma(j-k+1)} \\
& = \frac{(-1)^j 2^{2M+1} \Gamma(2M+1)}{\Gamma(2M-i+1) \Gamma(2M+i+2)} \sum_{k=0}^j \frac{(-j)_k (j+1)_k (2M+1)_k}{(2M-i+1)_k (2M+i+2)_k} \cdot \frac{1}{k!} \\
& = \frac{(-1)^j 2^{2M+1} \Gamma(2M+1)}{\Gamma(2M-i+1) \Gamma(2M+i+2)} {}_3F_2 \left(\begin{matrix} -j, j+1, 2M+1 \\ 2M-i+1, 2M+i+2 \end{matrix} \middle| 1 \right), \tag{3.8}
\end{aligned}$$

where $(a)_k$ is Pochhammer's symbol, defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} \quad (a \neq 0, -1, -2, \dots).$$

Here, we present Whipple's theorem concerning the hypergeometric series ${}_3F_2$.

Theorem 3.4 (Whipple's theorem [14])

$${}_3F_2 \left(\begin{matrix} a, 1-a, b \\ 1+2b-c, c \end{matrix} \middle| 1 \right) = \frac{\pi 2^{1-2b} \Gamma(c) \Gamma(1+2b-c)}{\Gamma(\frac{a+c}{2}) \Gamma(\frac{a+1+2b-c}{2}) \Gamma(\frac{1-a+c}{2}) \Gamma(\frac{2+2b-a-c}{2})}.$$

Setting $a = -j$, $b = 2M+1$, and $c = 2M-i+1$ in Theorem 3.4, we have

$$\begin{aligned}
& {}_3F_2 \left(\begin{matrix} -j, j+1, 2M+1 \\ 2M-i+1, 2M+i+2 \end{matrix} \middle| 1 \right) \\
& = \frac{\pi \Gamma(2M-i+1) \Gamma(2M+i+2)}{2^{4M+1} \Gamma(\frac{1}{2} - \frac{j}{2} - \frac{i}{2} + M) \Gamma(M + \frac{j}{2} - \frac{i}{2} + 1) \Gamma(M - \frac{j}{2} + \frac{i}{2} + 1) \Gamma(M + \frac{j}{2} + \frac{i}{2} + \frac{3}{2})}.
\end{aligned}$$

Substituting the above expression into (3.8), we obtain γ_{ij} in a closed form, as follows:

$$\gamma_{ij} = \frac{(-1)^{j+M} \pi \Gamma(2M+1) \sqrt{i + \frac{1}{2}} \sqrt{j + \frac{1}{2}}}{2^{2M} \Gamma(\frac{2M-i-j+1}{2}) \Gamma(\frac{2M+i-j+2}{2}) \Gamma(\frac{2M-i+j+2}{2}) \Gamma(\frac{2M+i+j+3}{2})} \quad (i-j: \text{even}).$$

This proves Theorem 3.3. □

The following statement follows from Theorems 3.2 and 3.3.

Corollary 3.1 $\psi_j(x)$ is a polynomial of $(2M+j)$ th degree and satisfies $\psi_j(-x) = (-1)^j \psi_j(x)$. Moreover, $\psi_j(x)$ is expanded by $\{\varphi_i(x)\}$, as follows:

$$\psi_{2i}(x) = \sum_{j=0}^{M+i} \gamma_{2i,2j} \varphi_{2j}(x), \quad \psi_{2i+1}(x) = \sum_{j=0}^{M+i} \gamma_{2i+1,2j+1} \varphi_{2j+1}(x).$$

Remark 3.1 It is often more convenient to express $G(x, y)$ in the following equivalent form:

$$\begin{aligned} G(x, y) = & G_0(x, y) - \sum_{i=0}^{\lfloor \frac{M-1}{2} \rfloor} \sum_{j=0}^{M+i} \gamma_{2i,2j} [\varphi_{2i}(x)\varphi_{2j}(y) + \varphi_{2i}(y)\varphi_{2j}(x)] \\ & - \sum_{i=0}^{\lfloor \frac{M-2}{2} \rfloor} \sum_{j=0}^{M+i} \gamma_{2i+1,2j+1} [\varphi_{2i+1}(x)\varphi_{2j+1}(y) + \varphi_{2i+1}(y)\varphi_{2j+1}(x)] \\ & + \sum_{i,j=0}^{\lfloor \frac{M-1}{2} \rfloor} \gamma_{2i,2j} \varphi_{2i}(x)\varphi_{2j}(y) + \sum_{i,j=0}^{\lfloor \frac{M-2}{2} \rfloor} \gamma_{2i+1,2j+1} \varphi_{2i+1}(x)\varphi_{2j+1}(y) \\ & (-1 < x, y < 1), \end{aligned}$$

where

$$\begin{aligned} \gamma_{2i,2j} = & \frac{(-1)^M \pi \Gamma(2M+1) \sqrt{(4i+1)(4j+1)}}{2^{2M+1} \Gamma(M+i-j+1) \Gamma(M-i+j+1) \Gamma(M+i+j+\frac{3}{2}) \Gamma(M-i-j+\frac{1}{2})} \\ & (0 \leq i, j \leq \lfloor (M-1)/2 \rfloor), \\ \gamma_{2i+1,2j+1} = & \frac{(-1)^{M+1} \pi \Gamma(2M+1) \sqrt{(4i+3)(4j+3)}}{2^{2M+1} \Gamma(M+i-j+1) \Gamma(M-i+j+1) \Gamma(M+i+j+\frac{5}{2}) \Gamma(M-i-j-\frac{1}{2})} \\ & (0 \leq i, j \leq \lfloor (M-2)/2 \rfloor). \end{aligned}$$

Theorem 3.1 is a direct consequence of the following theorem, which states that $G(x, y)$ serves as the Green function of BVP(M).

Theorem 3.5 *The Green function $G(x, y)$ satisfies the following conditions:*

- (1) $(-1)^M \partial_x^{2M} G(x, y) = - \sum_{i=0}^{M-1} \varphi_i(x) \varphi_i(y) \quad (-1 < x, y < 1, x \neq y),$
- (2) $\partial_x^i G(x, y)|_{x=\pm 1} = 0 \quad (M \leq i \leq 2M-1, -1 < y < 1),$
- (3) $\partial_x^i G(x, y)|_{y=x-0} - \partial_x^i G(x, y)|_{y=x+0} = \begin{cases} 0 & (0 \leq i \leq 2M-2), \\ (-1)^M & (i = 2M-1) \end{cases} \quad (-1 < x < 1),$
- (4) $\partial_x^i G(x, y)|_{x=y+0} - \partial_x^i G(x, y)|_{x=y-0} = \begin{cases} 0 & (0 \leq i \leq 2M-2), \\ (-1)^M & (i = 2M-1) \end{cases} \quad (-1 < y < 1),$
- (5) $\int_{-1}^1 \varphi_i(x) G(x, y) dx = 0 \quad (0 \leq i \leq M-1, -1 < y < 1).$

The proof of the above theorem is given in [13].

4 Sobolev inequality

In this section, it is shown that the Green function $G(x, y)$ is a reproducing kernel for a set of Hilbert space $H = H(M)$ and its inner product $(\cdot, \cdot)_M$ introduced in Section 1. We also derive the Sobolev inequality from the reproducing relation.

Theorem 4.1 For any $u(x) \in H$, we have the following reproducing relation:

$$u(y) = (u(x), G(x, y))_M = \int_{-1}^1 u^{(M)}(x) \partial_x^M G(x, y) dx \quad (-1 \leq y \leq 1). \quad (4.1)$$

In particular, we have

$$G(y, y) = \int_{-1}^1 |\partial_x^M G(x, y)|^2 dx \quad (-1 \leq y \leq 1). \quad (4.2)$$

Proof For functions $u = u(x)$ and $v = v(x) = G(x, y)$ with y arbitrarily fixed in $-1 \leq y \leq 1$, we have

$$u^{(M)} v^{(M)} - u(-1)^M v^{(2M)} = \left(\sum_{j=0}^{M-1} (-1)^{M-1-j} u^{(j)} v^{(2M-1-j)} \right)'.$$

Integrating with respect to x on intervals $-1 < x < y$ and $y < x < 1$ and using the properties of $G(x, y)$ given in Theorem 3.5, we have

$$\begin{aligned} (u, v)_M &= \int_{-1}^1 u^{(M)}(x) v^{(M)}(x) dx \\ &= \int_{-1}^1 u(x) (-1)^M v^{(2M)}(x) dx \\ &\quad + \left[\sum_{j=0}^{M-1} (-1)^{M-1-j} u^{(j)}(x) v^{(2M-1-j)}(x) \right] \left\{ \left[\begin{array}{c} x=y-0 \\ x=-1 \end{array} \right] + \left[\begin{array}{c} x=1 \\ x=y+0 \end{array} \right] \right\} \\ &= \sum_{j=0}^{M-1} (-1)^{M-1-j} [u^{(j)}(1) v^{(2M-1-j)}(1) - u^{(j)}(-1) v^{(2M-1-j)}(-1)] \\ &\quad + \sum_{j=0}^{M-1} (-1)^{M-1-j} u^{(j)}(y) [v^{(2M-1-j)}(y-0) - v^{(2M-1-j)}(y+0)] \\ &= u(y). \end{aligned}$$

This proves (4.1). Equation (4.2) is shown by setting $u(x) = G(x, y)$ in (4.1). \square

Applying the Schwarz inequality to (4.1) and using (4.2), we have

$$|u(y)|^2 \leq \int_{-1}^1 |\partial_x^M G(x, y)|^2 dx \int_{-1}^1 |u^{(M)}(x)|^2 dx = G(y, y) \int_{-1}^1 |u^{(M)}(x)|^2 dx.$$

Taking the supremum of the above inequality with respect to $y \in [-1, 1]$ and setting

$$C_0 = \max_{|y| \leq 1} G(y, y),$$

we obtain the following Sobolev inequality:

$$\left(\sup_{|y| \leq 1} |u(y)| \right)^2 \leq C_0 \int_{-1}^1 |u^{(M)}(x)|^2 dx = C_0 (u, u)_M. \quad (4.3)$$

This inequality shows that $(\cdot, \cdot)_M$ is positive definite. Note that, in order to prove (4.3), the Schwarz inequality is required but not the 'positive definiteness' of the inner product.

Let y_0 be a value that attains the maximum of $G(y, y)$. Applying this inequality to $u(x) = G(x, y_0) \in H$, we have

$$\left(\sup_{|y| \leq 1} |G(y, y_0)| \right)^2 \leq C_0 \int_{-1}^1 |\partial_x^M G(x, y_0)|^2 dx = C_0^2.$$

Combining this expression and the trivial inequality:

$$C_0^2 = \{G(y_0, y_0)\}^2 \leq \left(\sup_{|y| \leq 1} |G(y, y_0)| \right)^2,$$

we have

$$C_0^2 \leq \left(\sup_{|y| \leq 1} |G(y, y_0)| \right)^2 \leq C_0 \int_{-1}^1 |\partial_x^M G(x, y_0)|^2 dx = C_0^2.$$

Hence, it is shown that the equality in (4.3) holds if $u(x) = G(x, y_0)$:

$$\left(\sup_{|y| \leq 1} |G(y, y_0)| \right)^2 = C_0 \int_{-1}^1 |\partial_x^M G(x, y_0)|^2 dx.$$

Remark 4.1 In the next section, it is shown that $y_0 = \pm 1$ for $M \leq 5$, which is expected to hold also for $M \geq 6$.

5 Supremum of Sobolev functional

In this section, we prove (1.2) in Theorem 1.1 in the case of $M = 5$. The cases of $M \leq 4$ are comparatively simple, and so we omit their proofs. From Theorem 3.1, the diagonal value of the Green function $G(x, y) = G(5; x, y)$ is given as

$$\begin{aligned} G(5; x, x) = & \frac{1}{429,083,246,592} (3,693 + 1,750,653x^2 - 16,760,772x^4 + 56,270,844x^6 \\ & - 78,971,346x^8 + 44,618,574x^{10} - 7,910,916x^{12} + 1,701,564x^{14} \\ & - 263,891x^{16} + 20,349x^{18}). \end{aligned}$$

Calculating $G(5; 1, 1) - G(5; x, x)$, we have

$$\begin{aligned} G(5; 1, 1) - G(5; x, x) = & \frac{1}{429,083,246,592} (1 - x^2)g(5; x^2), \\ g(5; x) = & 455,059 - 1,295,594x + 15,465,178x^2 - 40,805,666x^3 + 38,165,680x^4 \\ & - 6,452,894x^5 + 1,458,022x^6 - 243,542x^7 + 20,349x^8 \quad (0 < x < 1). \end{aligned}$$

It is sufficient to prove the following lemma.

Lemma 5.1

$$g(x) = g(5; x) \geq 0 \quad (0 \leq x \leq 1).$$

Proof We investigate the extremal values or zeros of $g(x), g'(x), \dots, g^{(4)}(x)$. Through straightforward calculations, we have

$$\begin{aligned} g'''(0) &= -244,833,996, g'''(1) = 414,624,768, \\ g^{(4)}(x) &= 77,520(11,816 - 9,989x + 6,771x^2 - 2,639x^3 + 441x^4) \\ &= 77,520\{1,827 + 9,989(1-x) + 4,132x^2 + 2,639x^2(1-x) + 441x^4\} > 0. \end{aligned}$$

Therefore, the equation $g'''(x) = 0$ ($0 < x < 1$) possesses a unique solution, $x = x_0 \in (0, 1)$, and $g''(x)$ takes its minimum at $x = x_0$. Next, we prove

$$g''(x_0) < 0.$$

We can find $\frac{30}{100} < x_0 < \frac{31}{100}$ from the intermediate value theorem. Dividing $g''(x)$ by $g'''(x)$, we have

$$\begin{aligned} g''(x) &= \frac{1}{1,512}(252x - 377)g'''(x) + \frac{5}{126}r_0(x), \\ r_0(x) &= r_0^+(x) - r_0^-(x), \\ r_0^+(x) &= 46,067,875x^4 + 5,261,460,050x^2 + 613,870,628x, \\ r_0^-(x) &= 526,780,700x^3 + 758,928,637. \end{aligned}$$

Setting $x = x_0$, we conclude $g''(x_0) < 0$ as

$$\begin{aligned} g''(x_0) &= \frac{5}{126}r_0(x_0), \\ r_0(x_0) &= r_0^+(x_0) - r_0^-(x_0) < r_0^+\left(\frac{31}{100}\right) - r_0^-\left(\frac{30}{100}\right) \\ &= -\frac{61,440,051,132,097}{800,000} < 0. \end{aligned}$$

Together with the fact that $g''(0), g''(1) > 0$, there exist x_1, x_2 satisfying $0 < x_1 < x_0 < x_2 < 1$ and $g''(x_1) = g''(x_2) = 0$. In addition, $g'(x)$ takes its maximal and minimal values at $x = x_1$ and $x = x_2$, respectively. Next, we show that

$$g'(x_2) > 0.$$

Direct calculation shows that $\frac{41}{100} < x_2 < \frac{42}{100}$. Dividing $g'(x)$ by $g''(x)$, we have

$$\begin{aligned} g'(x) &= \frac{1}{1,764}(252x - 377)g''(x) + \frac{1}{441}r_1(x), \\ r_1(x) &= r_1^+(x) - r_1^-(x), \\ r_1^+(x) &= 138,203,625x^5 + 26,307,300,250x^3 + 4,604,029,710x^2 + 2,343,829,099, \\ r_1^-(x) &= 1,975,427,625x^4 + 11,383,929,555x. \end{aligned}$$

Setting $x = x_2$, we have

$$\begin{aligned} g'(x_2) &= \frac{1}{441} r_1(x_2), \\ r_1(x_2) &= r_1^+(x_2) - r_1^-(x_2) > r_1^+\left(\frac{41}{100}\right) - r_1^-\left(\frac{42}{100}\right) \\ &= \frac{7,181,871,438,953,829}{80,000,000} > 0, \end{aligned}$$

from which it is concluded that

$$g'(x_1) > g'(x_2) > 0.$$

Since we have $g'(0) < 0$, there exists a unique $x_3 \in (0, 1)$ such that $g'(x_3) = 0$, at which $g(x)$ takes its minimum value. Finally, we prove that

$$g(x_3) > 0.$$

Direct calculation and the intermediate value theorem show that $\frac{5}{100} < x_3 < \frac{6}{100}$. Dividing $g(x)$ by $g'(x)$, we have

$$\begin{aligned} g(x) &= \frac{1}{2,016} (252x - 377)g'(x) + \frac{1}{1,008} r_2(x), \\ r_2(x) &= r_2^+(x) - r_2^-(x), \\ r_2^+(x) &= 46,067,875x^6 + 13,153,650,125x^4 + 3,069,353,140x^3 \\ &\quad + 4,687,658,198x + 214,480,003, \\ r_2^-(x) &= 790,171,050x^5 + 11,383,929,555x^2. \end{aligned}$$

Using the same procedures, we obtain

$$\begin{aligned} g(x_3) &= \frac{1}{1,008} r_2(x_3), \\ r_2(x_3) &= r_2^+(x_3) - r_2^-(x_3) > r_2^+\left(\frac{5}{100}\right) - r_2^-\left(\frac{6}{100}\right) \\ &= \frac{653,353,651,584,933,307}{1,600,000,000} > 0. \end{aligned}$$

This shows that

$$g(x) \geq g(x_3) > 0 \quad (0 \leq x \leq 1),$$

which completes the proof of Lemma 5.1. \square

Although the proof is straightforward and simple, the proof for general M remains incomplete. Even if $M \geq 6$, if it is possible to carefully analyze the increase and decrease on the function $G(M; x, x)$, we will be able to confirm that the maximum value is achieved on the boundary points $x = \pm 1$. However, a unified way to treat $G(M; 1, 1) - G(M; x, x)$ has not been established and therefore the positivity remains to be unproved.

6 $G(1, 1)$

The purpose of this section is to prove (1.3) in Theorem 1.1,

$$G(1, 1) = \frac{2^{2M-1}\Gamma(2M-1)\Gamma(2M+1)}{\Gamma(M)^2\Gamma(4M)}. \quad (6.1)$$

Setting $y = 1$ in (4.2), we have

$$G(1, 1) = \int_{-1}^1 (\partial_x^M G(x, 1))^2 dx. \quad (6.2)$$

We present the following key lemma.

Lemma 6.1

$$\begin{aligned} \partial_x^M G(x, 1) &= \frac{\Gamma(M+1)}{2^M} K_M(1-x) K_{M-1}(1+x) \\ &= \frac{1}{2^M \Gamma(M)} (1-x)^{M-1} (1+x)^M \quad (-1 < x < 1). \end{aligned}$$

From the above lemma and (6.2), (6.1) is proven by

$$\begin{aligned} G(1, 1) &= \frac{1}{2^{2M} \Gamma(M)^2} \int_{-1}^1 (1-x)^{2M-2} (1+x)^{2M} dx \\ &= \frac{2^{2M-1}}{\Gamma(M)^2} B(2M-1, 2M+1) = \frac{2^{2M-1} \Gamma(2M-1) \Gamma(2M+1)}{\Gamma(M)^2 \Gamma(4M)}. \end{aligned}$$

Proof It is easy to see that

$$\nu(x) = \frac{\Gamma(M+1)}{2^M} K_M(1-x) K_{M-1}(1+x)$$

satisfies the boundary value problem:

$$\begin{cases} (-1)^{M-1} \nu^{(2M-1)} = \frac{\Gamma(2M)}{2^M \Gamma(M)} & (-1 < x < 1), \\ \nu^{(i)}(1) = 0 & (0 \leq i \leq M-2), \\ \nu^{(i)}(-1) = 0 & (0 \leq i \leq M-1). \end{cases} \quad (6.3)$$

$$(6.4)$$

$$(6.5)$$

Hence, it is sufficient to prove that

$$u(x) = \partial_x^M G(x, 1) \quad (-1 < x < 1)$$

satisfies the same boundary value problem (6.3)-(6.5).

From (3.1), we have

$$\begin{aligned} G(x, 1) &= \frac{(-1)^M}{2} K_0(1-x) - \sum_{i=0}^{M-1} \{ \psi_i(x) \varphi_i(1) + \psi_i(1) \varphi_i(x) \} \\ &\quad + \sum_{i,j=0}^{M-1} \gamma_{ij} \varphi_i(x) \varphi_j(1). \end{aligned}$$

Using $\partial_x^M \varphi_i(x) = 0$ ($-1 < x < 1$, $0 \leq i \leq M-1$), we have

$$u(x) = \partial_x^M G(x, 1) = \frac{1}{2} K_M(1-x) - \sum_{i=0}^{M-1} \varphi_i(1) \psi_i^{(M)}(x). \quad (6.6)$$

Recall the relation $(-1)^M \psi_i^{(2M)}(x) = \varphi_i(x)$ ($-1 < x < 1$), which follows from (3.2). We have

$$\begin{aligned} (-1)^{M-1} u^{(2M-1)}(x) &= (-1)^M \sum_{i=0}^{M-1} \varphi_i(1) \psi_i^{(3M-1)}(x) = \sum_{i=0}^{M-1} \varphi_i(1) \varphi_i^{(M-1)}(x) \\ &= \varphi_{M-1}(1) \varphi_{M-1}^{(M-1)}(x) = \left(M - \frac{1}{2}\right) P_{M-1}^{(M-1)}(x) \\ &= \frac{\Gamma(2M)}{2^M \Gamma(M)}. \end{aligned} \quad (6.7)$$

This shows that $u(x) = \partial_x^M G(x, 1)$ satisfies the differential equation (6.3).

Next, we prove that $u(x)$ satisfies the same boundary conditions (6.4) and (6.5). From (6.6), we have

$$u^{(k)}(x) = (-1)^k \frac{1}{2} K_{M+k}(1-x) - \sum_{i=0}^{M-1} \varphi_i(1) \psi_i^{(M+k)}(x).$$

We now prove the following lemma.

Lemma 6.2 *For any k ($0 \leq k \leq M-2$), we have*

$$(1) \quad u^{(k)}(1) = - \sum_{i=0}^{M-1} \varphi_i(1) \psi_i^{(M+k)}(1) = 0.$$

For any k ($0 \leq k \leq M-1$), we have

$$(2) \quad u^{(k)}(-1) = (-1)^k \frac{1}{2} K_{M+k}(2) - \sum_{i=0}^{M-1} \varphi_i(1) \psi_i^{(M+k)}(-1) = 0.$$

Proof We first prove (1). Substituting

$$\psi_i^{(M+k)}(1) = \int_{-1}^1 \frac{(-1)^M}{2} K_{M+k}(1-y) \varphi_i(y) dy \quad (0 \leq k \leq M-2)$$

into (1), we have

$$\sum_{i=0}^{M-1} \varphi_i(1) \psi_i^{(M+k)}(1) = \int_{-1}^1 \frac{(-1)^M}{2} K_{M+k}(1-y) \sum_{i=0}^{M-1} \varphi_i(1) \varphi_i(y) dy.$$

Since a vector space spanned by

$$\{\Gamma(M-k) K_{M+k}(1-y) = (1-y)^{M-1-k} \mid 0 \leq k \leq M-2\}$$

is the same as the vector space spanned by

$$\{\varphi_j(y) - \varphi_j(1) | 1 \leq j \leq M-1\},$$

it is sufficient to show that

$$\int_{-1}^1 \{\varphi_j(y) - \varphi_j(1)\} \sum_{i=0}^{M-1} \varphi_i(1) \varphi_i(y) dy = 0 \quad (1 \leq j \leq M-1).$$

This can be shown by

$$\begin{aligned} (\text{l.h.s.}) &= \int_{-1}^1 \left\{ \varphi_j(y) - \frac{\varphi_j(1)}{\varphi_0(1)} \varphi_0(y) \right\} \sum_{i=0}^{M-1} \varphi_i(1) \varphi_i(y) dy \\ &= \sum_{i=0}^{M-1} \left\{ \varphi_i(1) \int_{-1}^1 \varphi_j(y) \varphi_i(y) dy - \frac{\varphi_j(1) \varphi_i(1)}{\varphi_0(1)} \int_{-1}^1 \varphi_0(y) \varphi_i(y) dy \right\} = 0, \end{aligned}$$

where we used the following relations:

$$\varphi_0(y) = \varphi_0(1) = \frac{1}{\sqrt{2}} \quad \text{and} \quad \int_{-1}^1 \varphi_i(y) \varphi_j(y) dy = \delta_{ij}.$$

We next show (2), that is,

$$\sum_{i=0}^{M-1} \varphi_i(1) \psi_i^{(M+k)}(-1) = \frac{(-1)^k}{2} K_{M+k}(2) \quad (0 \leq k \leq M-1).$$

Since

$$\psi_i^{(M+k)}(-1) = \frac{(-1)^k}{2} \int_{-1}^1 K_{M+k}(1+y) \varphi_i(y) dy \quad (0 \leq k \leq M-1)$$

holds, it is sufficient to show that

$$\int_{-1}^1 K_{M+k}(1+y) \sum_{i=0}^{M-1} \varphi_i(1) \varphi_i(y) dy = K_{M+k}(2) \quad (0 \leq k \leq M-1),$$

or equivalently

$$\int_{-1}^1 \{K_{M+k}(2) - K_{M+k}(1+y)\} \sum_{i=0}^{M-1} \varphi_i(1) \varphi_i(y) dy = 0 \quad (0 \leq k \leq M-1).$$

This is easily confirmed in the case in which $k = M-1$, and therefore it is sufficient to prove this for $0 \leq k \leq M-2$. The vector space spanned by

$$\begin{aligned} &\{\Gamma(M-k)(K_{M+k}(2) - K_{M+k}(1+y)) = 2^{M-1-k} - (1+y)^{M-1-k} | \\ &0 \leq k \leq M-2\} \end{aligned}$$

is the same as the vector space spanned by

$$\{\Gamma(M-k)K_{M+k}(1-y) = (1-y)^{M-1-k} | 0 \leq k \leq M-2\}.$$

Using the same procedure as in the proof of (1), we can prove (2). \square

From (6.7) and Lemma 6.2, we have Lemma 6.1. \square

7 Infimum of Sobolev functional

In this section, we finally prove (1.4). It is sufficient to prove the following lemma.

Lemma 7.1 *The Legendre polynomials $P_n(x)$ ($n = 0, 1, 2, \dots$) satisfy the following properties:*

$$(1) \quad \sup_{|y| \leq 1} |P_n(y)| = 1,$$

$$(2) \quad \int_{-1}^1 |P_n^{(M)}(x)|^2 dx \rightarrow \infty \quad (n \rightarrow \infty, M = 1, 2, \dots).$$

Proof (1) follows from the Laplace-Mehler formula,

$$P_n(\cos \theta) = \frac{1}{\pi} \int_0^\pi (\cos \theta + \sqrt{-1} \sin \theta \cos \varphi)^n d\varphi,$$

and $P_n(1) = 1$. In order to prove (2), we present two lemmas.

Lemma 7.2

$$P_n^{(M)}(1) = \frac{\Gamma(n+M+1)}{2^M \Gamma(M+1) \Gamma(n-M+1)} \quad (n \geq M). \quad (7.1)$$

Proof $u = P_n(x)$ satisfies the Legendre differential equation:

$$(1-x^2)u'' - 2xu' + n(n+1)u = 0.$$

By taking M times the derivative of the above equation, it is shown that $v = v(x) = P_n^{(M)}(x)$ satisfies

$$(1-x^2)v'' - 2(M+1)xv' + \{n(n+1) - M(M+1)\}v = 0. \quad (7.2)$$

Setting $x = 1$ in (7.2), we have

$$-2(M+1)P_n^{(M+1)}(1) + \{n(n+1) - M(M+1)\}P_n^{(M)}(1) = 0.$$

This is equivalently rewritten as

$$\begin{aligned} \frac{\Gamma(M+2)\Gamma(n-M)}{\Gamma(n+M+2)} 2^{M+1} P_n^{(M+1)}(1) &= \frac{\Gamma(M+1)\Gamma(n-(M-1))}{\Gamma(n+M+1)} 2^M P_n^{(M)}(1) \\ &= \frac{\Gamma(1)\Gamma(n+1)}{\Gamma(n+1)} P_n(1) = 1. \end{aligned}$$

This proves Lemma 7.2. \square

Lemma 7.3 For $n \geq M$, we have

$$\int_{-1}^1 |P_n^{(M)}(x)|^2 dx \geq \frac{M}{n^2 + n - M^2} \left(\frac{\Gamma(n+M+1)}{2^M \Gamma(M+1) \Gamma(n-M+1)} \right)^2.$$

Proof Multiplying $v = P_n^{(M)}(x)$ on both sides of (7.2), we have

$$(1-x^2)v''v - 2(M+1)xv'v + \{n(n+1) - M(M+1)\}v^2 = 0,$$

or equivalently

$$(n^2 + n - M^2)v^2 = [Mxv^2 - (1-x^2)v'v]' + (1-x^2)(v')^2.$$

Integrating the above equality on the interval $-1 < x < 1$, we have

$$\begin{aligned} (n^2 + n - M^2) \int_{-1}^1 |v(x)|^2 dx &= M(v(1)^2 + v(-1)^2) + \int_{-1}^1 (1-x^2) |v'(x)|^2 dx \\ &\geq Mv(1)^2 = M(P_n^{(M)}(1))^2. \end{aligned}$$

Substituting (7.1) in the right-hand side of the above inequality, we obtain Lemma 7.3. \square

Taking the limit $n \rightarrow \infty$ in Lemma 7.3, we finally prove Lemma 7.1(2). \square

Proof of Theorem 1.3 From Lemma 7.1, we have

$$S(M; P_n(x)) = \left(\sup_{|y| \leq 1} |P_n(y)| \right)^2 / \int_{-1}^1 |P_n^{(M)}(x)|^2 dx \rightarrow 0 \quad (n \rightarrow \infty).$$

This proves (1.4). \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

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